

Perturbative renormalization factors of three- and four-quark operators for domain-wall QCD

¹Sinya Aoki, ¹Taku Izubuchi, ²Yoshinobu Kuramashi * and ¹Yusuke Taniguchi

¹*Institute of Physics, University of Tsukuba, Tsukuba, Ibaraki 305-8571, Japan*

²*Department of Physics, Washington University, St. Louis, Missouri 63130, USA*

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Abstract

Renormalization factors for three- and four-quark operators, which appear in the low energy effective Lagrangian of the proton decay and the weak interactions, are perturbatively calculated in domain-wall QCD. We find that the operators are multiplicatively renormalizable up to one-loop level without mixing with any other operators that have different chiral structures. As an application, we evaluate a renormalization factor for B_K at the parameters where previous simulations have been performed, and find one-loop corrections to B_K are 1-5% in these cases.

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*On leave from Institute of Particle and Nuclear Studies, High Energy Accelerator Research Organization(KEK), Tsukuba, Ibaraki 305-0801, Japan

I. INTRODUCTION

Calculation of hadron matrix elements of phenomenological interest represents an inevitable application of lattice QCD. In the past decade much efforts have been devoted for the calculation of three- and four-quark hadron matrix elements relevant to the proton decay amplitude and the weak interaction ones using the Wilson and the Kogut-Susskind(KS) quark actions. However, the satisfactorily precise measurement of the matrix elements has not been achieved so far because of the inherent defects in these quark actions: the explicit chiral symmetry breaking in the Wilson quark action causes the non-trivial operator mixing between different chiralities and for the KS quark it is hard to treat the heavy-light cases due to the flavor symmetry breaking.

The domain-wall quark formulation in lattice QCD, which is based on the introduction of many heavy regulator fields, was proposed by Shamir [1,2] anticipating superior features over other quark formulations: no need of the fine tuning to realize the chiral limit and no restriction for the number of flavors. Recent simulation results seem to support the former feature non-perturbatively [3–5]. It is also perturbatively shown that the massless mode at the tree level still remains stable against the quantum correction [6]. These advantageous features fascinate us to the application of the domain-wall quark for calculation of the three- and four-quark hadron matrix elements.

In order to convert the matrix elements obtained by lattice simulations to those defined in some continuum renormalization scheme(*e.g.*, $\overline{\text{MS}}$), we must know the renormalization factors connecting the lattice operators to the continuum counterparts defined in some renormalization scheme. In this article we make a perturbative calculation of the renormalization factors for the three- and four-quark operators consisting of physical quark fields in the domain-wall QCD(DWQCD). This work is an extension of the previous paper [7], in which we developed a perturbative renormalization procedure for DWQCD demonstrating the calculation of the renormalization factors for quark wave function, mass and bilinear operators. We focus on whether or not the renormalization of the three- and four-quark operators in DWQCD is free from the notorious operator mixing problem. In the Wilson case it is well known that the mixing problem is not adequately manipulated by the perturbation theory, leading to an “incorrect” value for the B_K matrix element.

This paper is organized as follows. In Sec. II we briefly introduce the DWQCD action and the Feynman rules relevant for the present calculation to make this paper self-contained. In Sec. III our calculational procedure of the renormalization factors for the four-quark operators is described in detail. We also evaluate the renormalization factors for three-quark operators in Sec. IV. In Secs. III and IV numerical results for one-loop coefficients of the renormalization factors are given with and without the mean field improvement. In Sec. V, using our results, we analyze a renormalization factor for B_K . Our conclusions are summarized in Sec. VI.

The physical quantities are expressed in lattice units and the lattice spacing a is suppressed unless necessary. We take $\text{SU}(N)$ gauge group with the gauge coupling g and the second Casimir $C_F = \frac{N^2 - 1}{2N}$, while $N = 3$ is specified in the numerical calculations.

II. ACTION AND FEYNMAN RULES

We take the Shamir's domain-wall fermion action [1],

$$\begin{aligned}
S_{\text{DW}} = & \sum_n \sum_{s=1}^{N_s} \left[\frac{1}{2} \sum_{\mu} \left(\bar{\psi}(n)_s (-r + \gamma_{\mu}) U_{\mu}(n) \psi(n + \mu)_s + \bar{\psi}(n)_s (-r - \gamma_{\mu}) U_{\mu}^{\dagger}(n - \mu) \psi(n - \mu)_s \right) \right. \\
& + \frac{1}{2} \left(\bar{\psi}(n)_s (1 + \gamma_5) \psi(n)_{s+1} + \bar{\psi}(n)_s (1 - \gamma_5) \psi(n)_{s-1} \right) + (M - 1 + 4r) \bar{\psi}(n)_s \psi(n)_s \Big] \\
& + m \sum_n \left(\bar{\psi}(n)_{N_s} P_+ \psi(n)_1 + \bar{\psi}(n)_1 P_- \psi(n)_{N_s} \right), \tag{1}
\end{aligned}$$

where n is a four dimensional space-time coordinate and s is an extra fifth dimensional or “flavor” index, the Dirac “mass” M is a parameter of the theory which we set $0 < M < 2$ to realize the massless fermion at tree level, m is a physical quark mass, and the Wilson parameter is set to $r = -1$. It is important to notice that we have boundaries for the flavor space; $1 \leq s \leq N_s$. In our one-loop calculation we will take $N_s \rightarrow \infty$ limit to avoid complications arising from the finite N_s . $P_{R/L}$ is a projection matrix $P_{R/L} = (1 \pm \gamma_5)/2$. For the gauge part we employ a standard four dimensional Wilson plaquette action and assume no gauge interaction along the fifth dimension.

In the DWQCD the zero mode of domain-wall fermion is extracted by the “physical” quark field defined by the boundary fermions

$$\begin{aligned}
q(n) &= P_R \psi(n)_1 + P_L \psi(n)_{N_s}, \\
\bar{q}(n) &= \bar{\psi}(n)_{N_s} P_R + \bar{\psi}(n)_1 P_L. \tag{2}
\end{aligned}$$

We will consider the QCD operators constructed from this quark fields, since this field has been actually used in the previous simulations. Moreover our renormalization procedure is based on the Green functions consisting of only the “physical” quark fields, in which we have found that the renormalization becomes simple [7].

Weak coupling perturbation theory is developed by expanding the action in terms of gauge coupling. The gluon propagator can be written as

$$G_{\mu\nu}^{AB}(k) = \delta_{\mu\nu} \delta_{AB} \frac{1}{4 \sin^2(k/2) + \lambda^2} \tag{3}$$

in the Feynman gauge with the infrared cut-off λ^2 , where $\sin^2(k/2) = \sum_{\mu} \sin^2(k_{\mu}/2)$. Quark-gluon vertices are also identical to those in the N_s flavor Wilson fermion. We need only one gluon vertex for our present calculation:

$$V_{1\mu}^A(k, p)_{st} = -igT^A \{ \gamma_{\mu} \cos(-k_{\mu}/2 + p_{\mu}/2) - ir \sin(-k_{\mu}/2 + p_{\mu}/2) \} \delta_{st}, \tag{4}$$

where k and p represent incoming momentum into the vertex (see Fig. 1 of Ref. [7]). T^A ($A = 1, \dots, N^2 - 1$) is a generator of color $\text{SU}(N)$.

The fermion propagator originally takes $N_s \times N_s$ matrix form in s -flavor space. In the present one-loop calculation, however, we do not need the whole matrix elements because we consider Green functions consisting of the physical quark fields. The relevant fermion propagators are restricted to following three types:

$$\langle q(-p)\bar{q}(p) \rangle = \frac{-i\gamma_\mu \sin p_\mu + (1 - We^{-\alpha})m}{-(1 - e^\alpha W) + m^2(1 - We^{-\alpha})} \equiv S_q(p), \quad (5)$$

$$\begin{aligned} \langle q(-p)\bar{\psi}(p, s) \rangle &= \frac{1}{F} \left(i\gamma_\mu \sin p_\mu - m(1 - We^{-\alpha}) \right) \left(e^{-\alpha(N_s-s)}P_R + e^{-\alpha(s-1)}P_L \right) \\ &+ \frac{1}{F} \left[m \left(i\gamma_\mu \sin p_\mu - m(1 - We^{-\alpha}) \right) - F \right] e^{-\alpha} \left(e^{-\alpha(s-1)}P_R + e^{-\alpha(N_s-s)}P_L \right), \end{aligned} \quad (6)$$

$$\begin{aligned} \langle \psi(-p, s)\bar{q}(p) \rangle &= \frac{1}{F} \left(e^{-\alpha(N_s-s)}P_L + e^{-\alpha(s-1)}P_R \right) \left(i\gamma_\mu \sin p_\mu - m(1 - We^{-\alpha}) \right) \\ &+ \frac{1}{F} \left(e^{-\alpha(s-1)}P_L + e^{-\alpha(N_s-s)}P_R \right) e^{-\alpha} \left[m \left(i\gamma_\mu \sin p_\mu - m(1 - We^{-\alpha}) \right) - F \right] \end{aligned} \quad (7)$$

with

$$W = 1 - M - r \sum_\mu (1 - \cos p_\mu), \quad (8)$$

$$\cosh(\alpha) = \frac{1 + W^2 + \sum_\mu \sin^2 p_\mu}{2|W|}, \quad (9)$$

$$F = 1 - e^\alpha W - m^2 (1 - We^{-\alpha}), \quad (10)$$

where the argument p in the factors α and W is suppressed.

In the perturbative calculation of Green functions the external quark momenta and masses are assumed to be much smaller than the lattice cut-off, so that we can expand the external quark propagators in terms of them. We have the following expressions as leading term of the expansion:

$$\langle q\bar{q} \rangle(p) = \frac{1 - w_0^2}{i\not{p} + (1 - w_0^2)m}, \quad (11)$$

$$\langle q\bar{\psi}_s \rangle(p) = \langle q\bar{q} \rangle(p) \left(w_0^{s-1}P_L + w_0^{N_s-s}P_R \right), \quad (12)$$

$$\langle \psi_s\bar{q} \rangle(p) = \left(w_0^{s-1}P_R + w_0^{N_s-s}P_L \right) \langle q\bar{q} \rangle(p), \quad (13)$$

where $w_0 = 1 - M$.

III. RENORMALIZATION FACTORS FOR FOUR-QUARK OPERATORS

We consider the following four-quark operators:

$$\mathcal{O}_\pm = \frac{1}{2} \left[(\bar{q}_1\gamma_\mu^L q_2)(\bar{q}_3\gamma_\mu^L q_4) \pm (\bar{q}_1\gamma_\mu^L q_4)(\bar{q}_3\gamma_\mu^L q_2) \right], \quad (14)$$

$$\mathcal{O}_1 = -C_F(\bar{q}_1\gamma_\mu^L q_2)(\bar{q}_3\gamma_\mu^R q_4) + (\bar{q}_1T^A\gamma_\mu^L q_2)(\bar{q}_3T^A\gamma_\mu^R q_4), \quad (15)$$

$$\mathcal{O}_2 = \frac{1}{2N}(\bar{q}_1\gamma_\mu^L q_2)(\bar{q}_3\gamma_\mu^R q_4) + (\bar{q}_1T^A\gamma_\mu^L q_2)(\bar{q}_3T^A\gamma_\mu^R q_4), \quad (16)$$

where $\gamma_\mu^{L,R} = \gamma_\mu P_{L,R}$. Summation over repeated indices such as μ and A is assumed. We note that q_i ($i = 1, 2, 3, 4$) are boundary quark fields in DWQCD. For the convenience of calculation we rewrite the above operators as

$$\mathcal{O}_\pm = \frac{1}{2} [1\tilde{\otimes}1 \pm 1\tilde{\odot}1]^{ab;cd} [(\bar{q}_1^a \gamma_\mu^L q_2^b)(\bar{q}_3^c \gamma_\mu^L q_4^d)], \quad (17)$$

$$\mathcal{O}_1 = \frac{1}{2} [-N1\tilde{\otimes}1 + 1\tilde{\odot}1]^{ab;cd} [(\bar{q}_1^a \gamma_\mu^L q_2^b)(\bar{q}_3^c \gamma_\mu^R q_4^d)], \quad (18)$$

$$\mathcal{O}_2 = \frac{1}{2} [1\tilde{\odot}1]^{ab;cd} [(\bar{q}_1^a \gamma_\mu^L q_2^b)(\bar{q}_3^c \gamma_\mu^R q_4^d)], \quad (19)$$

where a, b, c, d are color indices, and $\tilde{\otimes}, \tilde{\odot}$ represent the tensor structures in the color space:

$$[1\tilde{\otimes}1]^{ab;cd} \equiv \delta_{ab}\delta_{cd}, \quad (20)$$

$$[1\tilde{\odot}1]^{ab;cd} \equiv \delta_{ad}\delta_{cb}. \quad (21)$$

To derive these formula, we have used the Fierz transformation for \mathcal{O}_\pm and the formula

$$\sum_A T^A \tilde{\otimes} T^A = \frac{1}{2} \left[-\frac{1}{N} 1\tilde{\otimes}1 + 1\tilde{\odot}1 \right] \quad (22)$$

for $\mathcal{O}_{1,2}$.

We calculate the following Green function:

$$\langle \mathcal{O}_\Gamma \rangle_{\alpha\beta;\gamma\delta}^{ij;kl} \equiv \langle \mathcal{O}_\Gamma(q_1)_\alpha^i (\bar{q}_2)_\beta^j (q_3)_\gamma^k (\bar{q}_4)_\delta^l \rangle, \quad (23)$$

where $\Gamma = \pm, 1, 2$. Spinor indices are labeled by $\alpha, \beta, \gamma, \delta$ and color ones by i, j, k, l . Truncating the external quark propagators from $\langle \mathcal{O}_\Gamma \rangle$, where we multiply $\langle \mathcal{O}_\Gamma \rangle$ by $i\not{p}_i + (1 - w_0^2)m$, we obtain the vertex functions, which is written in the following form up to the one-loop level

$$(1 - w_0^2)^4 (\Lambda_\Gamma)_{\alpha\beta;\gamma\delta}^{ij;kl} = (1 - w_0^2)^4 \left(\Lambda_\Gamma^{(0)} + \Lambda_\Gamma^{(1)} \right)_{\alpha\beta;\gamma\delta}^{ij;kl}, \quad (24)$$

where the superscript (i) refers to the i -th loop level and the trivial factor $(1 - w_0^2)^4$ is factored out for the convenience. We suppress the external momenta p_i since the renormalization factor does not depend on them.

The tree level vertex functions $\Lambda_\Gamma^{(0)}$ are given by

$$\Gamma = \pm, \quad \frac{1}{2} [\gamma_\mu^L \otimes \gamma_\mu^L]_{\alpha\beta;\gamma\delta} [1\tilde{\otimes}1 \pm 1\tilde{\odot}1]^{ij;kl}, \quad (25)$$

$$\Gamma = 1, \quad \frac{1}{2} [\gamma_\mu^L \otimes \gamma_\mu^R]_{\alpha\beta;\gamma\delta} [-N1\tilde{\otimes}1 + 1\tilde{\odot}1]^{ij;kl}, \quad (26)$$

$$\Gamma = 2, \quad \frac{1}{2} [\gamma_\mu^L \otimes \gamma_\mu^R]_{\alpha\beta;\gamma\delta} [1\tilde{\odot}1]^{ij;kl}, \quad (27)$$

where \otimes acts on the Dirac spinor space representing $[\gamma_X \otimes \gamma_Y]_{\alpha\beta;\gamma\delta} \equiv (\gamma_X)_{\alpha\beta}(\gamma_Y)_{\gamma\delta}$.

The one-loop vertex corrections are illustrated by six diagrams in Fig. 1, the sum of which yields the one-loop level vertex function

$$\Lambda_\Gamma^{(1)} = \int_{-\pi}^{\pi} \frac{d^4 k}{(2\pi)^4} (I_\Gamma^a + \dots + I_\Gamma^{c'}). \quad (28)$$

In order to obtain the expressions for the integrands $I_\Gamma^a, \dots, I_\Gamma^{c'}$ we should note that the internal quark propagators appearing in the diagrams are multiplied by the damping factor which comes from eqs.(12) and (13). The following formula are useful.

$$\langle q\bar{\psi}_s \rangle \left(w_0^{s-1} P_L + w_0^{N_s-s} P_R \right) = \left(w_0^{s-1} P_R + w_0^{N_s-s} P_L \right) \langle \psi_s \bar{q} \rangle = \frac{i\gamma_\mu \sin p_\mu}{\tilde{F} \cdot \tilde{F}_0} \equiv \overline{G}, \quad (29)$$

$$\langle q\bar{\psi}_s \rangle \left(w_0^{s-1} P_R + w_0^{N_s-s} P_L \right) = \left(w_0^{s-1} P_L + w_0^{N_s-s} P_R \right) \langle \psi_s \bar{q} \rangle = -\frac{1}{\tilde{F}_0} \equiv \tilde{G}, \quad (30)$$

where $\tilde{F} = e^{-\alpha} - W$ and $\tilde{F}_0 = e^\alpha - w_0$. Here we set $m = p_i = 0$ for the internal propagator.

The contribution from Fig. 1a takes the form

$$I_\Gamma^a = \frac{1}{2} J_a^{AB} \left\{ \overline{V}_\mu(k) \overline{G}(k) + \tilde{V}_\mu(k) \tilde{G}(k) \right\} \Gamma_X \left\{ \overline{G}(k) \overline{V}_\nu(k) + \tilde{G}(k) \tilde{V}_\nu(k) \right\} \otimes \Gamma_Y G_{\mu\nu}^{AB}(k), \quad (31)$$

where $\Gamma_X = \gamma_\mu^L$, $\Gamma_Y = \gamma_\mu^L$ or γ_μ^R , and the interaction vertices are

$$\overline{V}_\mu = -ig\gamma_\mu \cos(k_\mu/2), \quad \tilde{V}_\mu = -rg \sin(k_\mu/2). \quad (32)$$

The color factors are represented by J_a^{AB} , which are listed in Table I. In a similar way the contributions from Fig. 1b and Fig. 1c are given by

$$I_\Gamma^b = \frac{1}{2} J_b^{AB} \left\{ \overline{V}_\mu(k) \overline{G}(k) + \tilde{V}_\mu(k) \tilde{G}(k) \right\} \Gamma_X \otimes \left\{ \overline{V}_\nu(-k) \overline{G}(-k) + \tilde{V}_\nu(-k) \tilde{G}(-k) \right\} \Gamma_Y G_{\mu\nu}^{AB}(k), \quad (33)$$

$$I_\Gamma^c = \frac{1}{2} J_c^{AB} \left\{ \overline{V}_\mu(k) \overline{G}(k) + \tilde{V}_\mu(k) \tilde{G}(k) \right\} \Gamma_X \otimes \Gamma_Y \left\{ \overline{G}(k) \overline{V}_\nu(k) + \tilde{G}(k) \tilde{V}_\nu(k) \right\} G_{\mu\nu}^{AB}(k). \quad (34)$$

After a little algebra the expressions of $I_\Gamma^{a,b,c}$ are reduced to

$$I_\Gamma^a = \frac{1}{2} g^2 J_a^{AA} K [T + A_{VA}] [\Gamma_X \otimes \Gamma_Y], \quad (35)$$

$$I_\Gamma^b = -\frac{1}{2} g^2 J_b^{AA} K [T \Gamma_X \otimes \Gamma_Y + \cos^2(k_\mu/2) \sin^2 k_\alpha (\gamma_\mu \gamma_\alpha \Gamma_X) \otimes (\gamma_\mu \gamma_\alpha \Gamma_Y)], \quad (36)$$

$$I_\Gamma^c = \frac{1}{2} g^2 J_c^{AA} K [T \Gamma_X \otimes \Gamma_Y + \cos^2(k_\mu/2) \sin^2 k_\alpha (\gamma_\mu \gamma_\alpha \Gamma_X) \otimes (\Gamma_Y \gamma_\alpha \gamma_\mu)], \quad (37)$$

where

$$K = \frac{1}{\tilde{F}^2 \tilde{F}_0^2 (4 \sin^2(k/2) + \lambda^2)}, \quad (38)$$

$$T = r^2 \sin^2(k/2) \tilde{F}^2 + r \sin^2 k \tilde{F}, \quad (39)$$

$$A_{VA} = \sum_\mu \cos^2(k_\mu/2) \sin^2 k_\mu. \quad (40)$$

In order to rewrite the second term of $I_\Gamma^{b,c}$ we apply the Fierz transformation:

$$(\gamma_\mu \gamma_\alpha \gamma_\nu^L) \otimes (\gamma_\mu \gamma_\alpha \gamma_\nu^L) = -\gamma_\nu^L \odot \gamma_\nu^L = \gamma_\nu^L \otimes \gamma_\nu^L, \quad (41)$$

$$(\gamma_\mu \gamma_\alpha \gamma_\nu^L) \otimes (\gamma_\nu^L \gamma_\alpha \gamma_\mu) = -\gamma_\nu^L \odot \gamma_\nu^L (1 - 2\delta_{\alpha\nu}) (1 - 2\delta_{\mu\nu}) \quad (42)$$

$$= -\gamma_\nu^L \otimes \gamma_\nu^L (1 - 2\delta_{\alpha\nu}) (1 - 2\delta_{\mu\nu}) + 2\gamma_\nu^L \otimes \gamma_\nu^L \delta_{\alpha\mu}, \quad (43)$$

$$(\gamma_\mu \gamma_\alpha \gamma_\nu^L) \otimes (\gamma_\mu \gamma_\alpha \gamma_\nu^R) = 2P_R \odot P_L \delta_{\mu\alpha} = \gamma_\nu^L \otimes \gamma_\nu^R \delta_{\mu\alpha}, \quad (44)$$

$$(\gamma_\mu \gamma_\alpha \gamma_\nu^L) \otimes (\gamma_\nu^R \gamma_\alpha \gamma_\mu) = 2P_R \odot P_L = \gamma_\nu^L \otimes \gamma_\nu^R. \quad (45)$$

where $[\gamma_X \odot \gamma_Y]_{\alpha\beta;\gamma\delta} \equiv (\gamma_X)_{\alpha\delta}(\gamma_Y)_{\gamma\beta}$, and summation over ν is taken. The Fierz transformation is again used for the second equality. We omit the tensor term in eq.(44) since it vanishes in the integral.

Choosing $\Gamma_{X,Y} = \gamma_\mu^L$ in eqs.(35), (36) and (37) we first consider the case of \mathcal{O}_\pm . After simplifying the expressions of the color factors we obtain

$$I_\pm^a = \frac{1}{2}g^2K(T + A_{VA}) [\gamma_\nu^L \otimes \gamma_\nu^L] \left[(C_F \pm \frac{1}{2})1\tilde{\otimes}1 \mp \frac{1}{2N}1\tilde{\odot}1 \right], \quad (46)$$

$$I_\pm^b = -\frac{1}{2}g^2K(T + A_{SP}) [\gamma_\nu^L \otimes \gamma_\nu^L] \left[(-\frac{1}{2N} \pm \frac{1}{2})1\tilde{\otimes}1 + (\frac{1}{2} \mp \frac{1}{2N})1\tilde{\odot}1 \right], \quad (47)$$

$$I_\pm^c = \frac{1}{2}g^2K(T + A_{VA}) [\gamma_\nu^L \otimes \gamma_\nu^L] \left[-\frac{1}{2N}1\tilde{\otimes}1 + (\frac{1}{2} \pm C_F)1\tilde{\odot}1 \right]. \quad (48)$$

The total contribution becomes

$$\begin{aligned} I_+^a + I_+^b + I_+^c &= \frac{1}{2}g^2K [\gamma_\nu^L \otimes \gamma_\nu^L]_{\alpha\beta;\gamma\delta} [1\tilde{\otimes}1 + 1\tilde{\odot}1]^{ij;kl} \\ &\times \{TC_F + A_{VA}(C_F + \frac{1}{2} - \frac{1}{2N}) - A_{SP}(\frac{1}{2} - \frac{1}{2N})\}, \end{aligned} \quad (49)$$

and

$$\begin{aligned} I_-^a + I_-^b + I_-^c &= \frac{1}{2}g^2K [\gamma_\nu^L \otimes \gamma_\nu^L]_{\alpha\beta;\gamma\delta} [1\tilde{\otimes}1 - 1\tilde{\odot}1]^{ij;kl} \\ &\times \{TC_F + A_{VA}(C_F - \frac{1}{2} - \frac{1}{2N}) + A_{SP}(\frac{1}{2} + \frac{1}{2N})\}, \end{aligned} \quad (50)$$

where

$$A_{SP} = \cos^2(k/2) \cdot \sin^2 k. \quad (51)$$

We should note that the other three contributions $I_\pm^{a',b',c'}$ from Fig. 1a', 1b' and 1c' are equal to $I_\Gamma^{a,b,c}$ respectively, therefore the factors 1/2 in eqs. (49) and (50) disappear in the total contributions of all.

Comparing the one-loop results to the tree level ones we obtain

$$\Lambda_+ = \left[1 + g^2 \frac{N-1}{N} \{ \langle T \rangle (N+1) + \langle A_{VA} \rangle (N+2) - \langle A_{SP} \rangle \} \right] \Lambda_+^{(0)}, \quad (52)$$

$$\Lambda_- = \left[1 + g^2 \frac{N+1}{N} \{ \langle T \rangle (N-1) + \langle A_{VA} \rangle (N-2) + \langle A_{SP} \rangle \} \right] \Lambda_-^{(0)} \quad (53)$$

with

$$\langle X \rangle = \int_{-\pi}^{\pi} \frac{d^4k}{(2\pi)^4} K(k) X(k) \quad (54)$$

for $X = T, A_{VA}, A_{SP}$. We remark that $C_F < T + A_{VA} >$ and $C_F < T + A_{SP} >$ correspond to the one-loop vertex corrections to the (axial) vector current and the (pseudo) scalar density which are expressed as $(T_{VA} - 1)$ and $(T_{SP} - 1)$ in Ref. [7]. The expressions of eqs.(52) and (53) show an important property of \mathcal{O}_\pm in the DWQCD formalism: the one-loop vertex

corrections are multiplicative. This is contrary to the Wilson case, in which the mixing operators with different chiralities appears at the one-loop level [8].

We next turn to the case of $\mathcal{O}_{1,2}$. For \mathcal{O}_1 the vertex corrections of eqs.(35), (36) and (37) with $\Gamma_X = \gamma_\mu^L$ and $\Gamma_Y = \gamma_\mu^R$ are written as

$$I_1^a = \frac{1}{2}g^2K(T + A_{VA}) [\gamma_\nu^L \otimes \gamma_\nu^R] \left[(-NC_F + \frac{1}{2})1\tilde{\otimes}1 - \frac{1}{2N}1\tilde{\odot}1 \right], \quad (55)$$

$$I_1^b = -\frac{1}{2}g^2K(T + A_{VA}) [\gamma_\nu^L \otimes \gamma_\nu^R] \left[(\frac{1}{2} + \frac{1}{2})1\tilde{\otimes}1 + (-\frac{N}{2} - \frac{1}{2N})1\tilde{\odot}1 \right], \quad (56)$$

$$I_1^c = \frac{1}{2}g^2K(T + A_{SP}) [\gamma_\nu^L \otimes \gamma_\nu^R] \left[\frac{1}{2}1\tilde{\otimes}1 + (-\frac{N}{2} + C_F)1\tilde{\odot}1 \right]. \quad (57)$$

The total contribution including those from Fig. 1a', 1b' and 1c' is given by

$$\begin{aligned} 2(I_1^a + I_1^b + I_1^c) &= 2\frac{1}{2}g^2K [\gamma_\nu^L \otimes \gamma_\nu^R]_{\alpha\beta;\gamma\delta} \left[-N1\tilde{\otimes}1 + 1\tilde{\odot}1 \right]^{ij;kl} \\ &\times \left[TC_F + A_{VA}\frac{N}{2} - A_{SP}\frac{1}{2N} \right]. \end{aligned} \quad (58)$$

Using the tree level result in eq.(26) the vertex function up to the one-loop level is expressed as

$$\Lambda_1 = \left[1 + g^2\frac{1}{N} \left\{ \langle T \rangle (N^2 - 1) + \langle A_{VA} \rangle N^2 - \langle A_{SP} \rangle \right\} \right] \Lambda_1^{(0)}. \quad (59)$$

This result shows that the operator \mathcal{O}_1 is multiplicatively renormalizable in DWQCD, which is in contrast with the Wilson case [8].

In a similar way we write the vertex corrections for \mathcal{O}_2 .

$$I_2^a = \frac{1}{2}g^2K(T + A_{VA}) [\gamma_\nu^L \otimes \gamma_\nu^R] \left[\frac{1}{2}1\tilde{\otimes}1 - \frac{1}{2N}1\tilde{\odot}1 \right], \quad (60)$$

$$I_2^b = -\frac{1}{2}g^2K(T + A_{VA}) [\gamma_\nu^L \otimes \gamma_\nu^R] \left[\frac{1}{2}1\tilde{\otimes}1 - \frac{1}{2N}1\tilde{\odot}1 \right], \quad (61)$$

$$I_2^c = \frac{1}{2}g^2K(T + A_{SP}) [\gamma_\nu^L \otimes \gamma_\nu^R] [C_F 1\tilde{\odot}1]. \quad (62)$$

The total contribution including those from Fig. 1a', 1b' and 1c' becomes

$$\begin{aligned} 2(I_2^a + I_2^b + I_2^c) &= 2\frac{1}{2}g^2K [\gamma_\nu^L \otimes \gamma_\nu^R]_{\alpha\beta;\gamma\delta} [1\tilde{\odot}1]^{ij;kl} \\ &\times C_F [T + A_{SP}], \end{aligned} \quad (63)$$

which leads to

$$\Lambda_2 = \left[1 + g^2\frac{N^2 - 1}{N} \left\{ \langle T \rangle + \langle A_{SP} \rangle \right\} \right] \Lambda_2^{(0)}. \quad (64)$$

We again find that the vertex correction is multiplicative up to the one-loop level as opposed to the Wilson case [8].

The contribution from the fermion self-energy has already been evaluated [6,7] and the total lattice renormalization factor is now obtained:

$$Z_{\Gamma}^{lat} = (1 - w_0^2)^2 Z_w^2 Z_2^2 V_{\Gamma}, \quad (65)$$

where

$$Z_2 = 1 + \frac{g^2}{16\pi^2} C_F [\log(\lambda a)^2 + \Sigma_1], \quad (66)$$

$$V_{\Gamma} = 1 + \frac{g^2}{16\pi^2} [-\delta_{\Gamma} \log(\lambda a)^2 + v_{\Gamma}], \quad (67)$$

$$v_+ = \frac{16\pi^2(N-1)}{N} [< T > (N+1) + << A_{VA} >> (N+2) - << A_{SP} >>] + \delta_+ \log \pi^2, \quad (68)$$

$$v_- = \frac{16\pi^2(N+1)}{N} [< T > (N-1) + << A_{VA} >> (N-2) + << A_{SP} >>] + \delta_- \log \pi^2, \quad (69)$$

$$v_1 = \frac{16\pi^2}{N} [< T > (N^2 - 1) + << A_{VA} >> N^2 - << A_{SP} >>] + \delta_1 \log \pi^2, \quad (70)$$

$$v_2 = \frac{16\pi^2(N^2 - 1)}{N} [< T > + << A_{SP} >>] + \delta_2 \log \pi^2 \quad (71)$$

with

$$\delta_{\Gamma} = \begin{cases} \frac{(N-1)(N-2)}{N} & \Gamma = + \\ \frac{(N+1)(N+2)}{N} & \Gamma = - \\ \frac{(N+2)(N-2)}{N} & \Gamma = 1 \\ \frac{4(N+1)(N-1)}{N} & \Gamma = 2 \end{cases}$$

The infrared singularity of $< A_X >$ is subtracted as

$$<< A_X >> = \int_{-\pi}^{\pi} \frac{d^4 k}{(2\pi)^4} \left[K(k) A_X(k) - c_X \frac{1}{(k^2)^2} \theta(\pi^2 - k^2) \right]$$

with $c_{SP} = 4$ and $c_{VA} = 1$.

Numerical values of v_{Γ} are evaluated by two independent methods. In one method the momentum integration is performed by a mode sum for a periodic box of a size L^4 after transforming the momentum variable through $k_{\mu} = q_{\mu} - \sin q_{\mu}$. We employ the size $L = 64$ for integrals. In the other method the momentum integration is carried out by the Monte Carlo integration routine VEGAS, using 20 samples of 1000000 points each. We find that both results agree very well. Numerical values of v_{Γ} are presented in Table II as a function of M .

We have to also calculate the corresponding continuum wave-function renormalization factor and vertex corrections in the $\overline{\text{MS}}$ scheme employing the same gauge and the same infrared regulator as the lattice case. For the present calculation it seems preferable to choose

Dimensional Reduction(DRED) as the ultraviolet regularization, in which the loop momenta of the Feynman integrals are defined in $D < 4$ dimensions while keeping the Dirac matrices in four dimensions. In the DRED scheme we can use the same calculational techniques for the vertex corrections as the lattice case thanks to applicability of the Fierz transformation for the Dirac matrices. For the wave-function renormalization factor a simple calculation gives

$$Z_2^{\overline{\text{MS}}} = 1 + \frac{g^2}{16\pi^2} C_F \left[\log(\lambda/\mu)^2 - 1/2 \right], \quad (72)$$

where μ is a renormalization scale. This result leads to $\Sigma_1^{\overline{\text{MS}}} = -1/2$. For the vertex corrections we obtain

$$V_\Gamma^{\overline{\text{MS}}} = 1 + \frac{g^2}{16\pi^2} \delta_\Gamma \left[-\log(\lambda/\mu)^2 + 1 \right], \quad (73)$$

giving $v_\Gamma^{\overline{\text{MS}}} = \delta_\Gamma$. Here we should remark that the one-loop vertex corrections yield the evanescent operators which vanish in $D = 4$ for the DRED scheme [9]. It is meaningless to give results without mentioning the definition of evanescent operators, because the constant terms at the one-loop level depend on the definition of the evanescent operators. Our choice is as follows:

$$E_\pm^{\text{DRED}} = \bar{\delta}_{\mu\nu} \gamma_\mu (1 - \gamma_5) \otimes \gamma_\nu (1 - \gamma_5) - \frac{D}{4} \gamma_\mu (1 - \gamma_5) \otimes \gamma_\mu (1 - \gamma_5), \quad (74)$$

$$E_{1,2}^{\text{DRED}} = \bar{\delta}_{\mu\nu} \gamma_\mu (1 - \gamma_5) \otimes \gamma_\nu (1 + \gamma_5) - \frac{D}{4} \gamma_\mu (1 - \gamma_5) \otimes \gamma_\mu (1 + \gamma_5), \quad (75)$$

where $\bar{\delta}_{\mu\nu}$ is the D -dimensional metric tensor which emerges inevitably in the evaluation of the Feynman integrals.

Combining these results with the previous lattice ones we obtain

$$\mathcal{O}_\Gamma^{\overline{\text{MS}}}(\mu) = \frac{1}{(1 - w_0^2)^2 Z_w^2} Z_\Gamma(\mu a) \mathcal{O}_\Gamma^{\text{lat}}(1/a), \quad (76)$$

where

$$\begin{aligned} Z_\Gamma(\mu a) &= \frac{(Z_2^{\overline{\text{MS}}})^2 V_\Gamma^{\overline{\text{MS}}}}{(Z_2)^2 V_\Gamma} \\ &= 1 + \frac{g^2}{16\pi^2} \left[(\delta_\Gamma - 2C_F) \log(\mu a)^2 + z_\Gamma \right], \end{aligned} \quad (77)$$

$$z_\Gamma = v_\Gamma^{\overline{\text{MS}}} - v_\Gamma + 2C_F \{ \Sigma_1^{\overline{\text{MS}}} - \Sigma_1 \}. \quad (78)$$

Numerical values of z_Γ are given in Table III and the results for the mean-field improved one, z_Γ^{MF} , are also given in Table IV.

Although the results for the DRED scheme are presented here, it is an easy task to obtain those for the Naive Dimensional Regularization(NDR) scheme. In Appendix B we summarize the finite parts of the wave-function renormalization factor and vertex corrections in the NDR scheme.

IV. RENORMALIZATION FACTORS FOR THREE-QUARK OPERATORS

The three-quark operators relevant to the proton decay amplitude are given by

$$(\mathcal{O}_{PD})_\delta = \varepsilon^{abc} \left((\bar{q}_1^C)^a \Gamma_X(q_2)^b \right) (\Gamma_Y(q_3)^c)_\delta, \quad (79)$$

where $\bar{q}^C = -q^T C^{-1}$ with $C = \gamma_0 \gamma_2$ is a charge conjugated field of q and $\Gamma_X \otimes \Gamma_Y = P_R \otimes P_R, P_R \otimes P_L, P_L \otimes P_R, P_L \otimes P_L$. The summation over repeated color indices a, b, c is assumed. We should note that the domain-wall fermion action (1) is transformed identically into that with the conjugated field by using transpose and matrix C . The resultant action and Feynman rules for the conjugated field is obtained by the replacement that

$$igT^A \rightarrow -ig(T^A)^T, \quad (80)$$

where the superscript T means the transposed matrix.

In order to evaluate the vertex corrections we consider the following Green function:

$$\langle (\mathcal{O}_{PD})_\delta \rangle_{\alpha\beta\gamma}^{ijk} \equiv \langle (\mathcal{O}_{PD})_\delta (q_1^C)_\alpha^i (\bar{q}_2)_\beta^j (\bar{q}_3)_\gamma^k \rangle, \quad (81)$$

where α, β, γ and i, j, k are spinor and color indices respectively. Truncating the external quark propagators of $\langle \mathcal{O}_{PD} \rangle$ we obtain the vertex function

$$(1 - w_0^2)^3 (\Lambda_{PD})_{\alpha\beta;\delta\gamma}^{ijk} = (1 - w_0^2)^3 \left(\Lambda_\Gamma^{(0)} + \Lambda_\Gamma^{(1)} \right)_{\alpha\beta;\delta\gamma}^{ijk}, \quad (82)$$

where the trivial factor $(1 - w_0^2)^3$ is factored out for the convenience. We suppress the external momenta p_i since the renormalization factor does not depend on them.

At the tree level the vertex function takes the form

$$\Lambda_\Gamma^{(0)} = \varepsilon^{ijk} [\Gamma_X \otimes \Gamma_Y]_{\alpha\beta;\delta\gamma}, \quad (83)$$

where $[\Gamma_X \otimes \Gamma_Y]_{\alpha\beta;\delta\gamma} \equiv (\Gamma_X)_{\alpha\beta} (\Gamma_Y)_{\delta\gamma}$.

The one-loop vertex corrections are shown in Figs. 2a, 2b and 2c, the sum of which gives the one-loop level vertex function

$$\Lambda_{PD}^{(1)} = \int_{-\pi}^{\pi} \frac{d^4 k}{(2\pi)^4} \left(I_{PD}^a + I_{PD}^b + I_{PD}^c \right). \quad (84)$$

Using the notations in eqs.(29), (30) and (32) the integrands $I_{PD}^{a,b,c}$ are written as follows:

$$I_{PD}^a = \varepsilon^{abk} (-T^A)_{ia}^T T_{bj}^B \times \left\{ \bar{V}_\mu(k) \bar{G}(k) + \tilde{V}_\mu(k) \tilde{G}(k) \right\} \Gamma_X \left\{ \bar{G}(k) \bar{V}_\nu(k) + \tilde{G}(k) \tilde{V}_\nu(k) \right\} \otimes \Gamma_Y G_{\mu\nu}^{AB}(k), \quad (85)$$

$$I_{PD}^b = \varepsilon^{ibc} T_{bj}^A T_{ck}^B \times \Gamma_X \left\{ \bar{G}(k) \bar{V}_\nu(k) + \tilde{G}(k) \tilde{V}_\nu(k) \right\} \otimes \Gamma_Y \left\{ \bar{G}(-k) \bar{V}_\nu(-k) + \tilde{G}(-k) \tilde{V}_\nu(-k) \right\} G_{\mu\nu}^{AB}(k), \quad (86)$$

$$I_{PD}^c = \varepsilon^{ajc} (-T^A)_{ia}^T T_{ck}^B \times \left\{ \bar{V}_\mu(k) \bar{G}(k) + \tilde{V}_\mu(k) \tilde{G}(k) \right\} \Gamma_X \otimes \Gamma_Y \left\{ \bar{G}(k) \bar{V}_\nu(k) + \tilde{G}(k) \tilde{V}_\nu(k) \right\} G_{\mu\nu}^{AB}(k). \quad (87)$$

A little algebra yields

$$I_{PD}^a = g^2 \frac{N+1}{2N} \varepsilon^{ijk} K [T + A_{SP}] [\Gamma_X \otimes \Gamma_Y], \quad (88)$$

$$I_{PD}^b = g^2 \frac{N+1}{2N} \varepsilon^{ijk} K \left[T(\Gamma_X \otimes \Gamma_Y) + \cos^2(k_\mu/2) \sin^2 k_\alpha (\Gamma_X \gamma_\alpha \gamma_\mu) \otimes (\Gamma_Y \gamma_\alpha \gamma_\mu) \right], \quad (89)$$

$$I_{PD}^c = g^2 \frac{N+1}{2N} \varepsilon^{ijk} K \left[T(\Gamma_X \otimes \Gamma_Y) + \cos^2(k_\mu/2) \sin^2 k_\alpha (\gamma_\mu \gamma_\alpha \Gamma_X) \otimes (\Gamma_Y \gamma_\alpha \gamma_\mu) \right], \quad (90)$$

where K , T and A_{SP} are given in eqs.(38), (39) and (51). It is noted that a sum of I_{PD}^b and I_{PD}^c becomes

$$\begin{aligned} & g^2 \frac{N+1}{2N} \varepsilon^{ijk} K \left[2T(\Gamma_X \otimes \Gamma_Y) + \cos^2(k_\mu/2) \sin^2 k_\alpha (\gamma_\mu \gamma_\alpha \Gamma_X + \Gamma_X \gamma_\alpha \gamma_\mu) \otimes (\Gamma_Y \gamma_\alpha \gamma_\mu) \right] \\ &= g^2 \frac{N+1}{2N} \varepsilon^{ijk} K \left[2T(\Gamma_X \otimes \Gamma_Y) + \cos^2(k_\mu/2) \sin^2 k_\alpha (\{\gamma_\mu \gamma_\alpha + \gamma_\alpha \gamma_\mu\} \Gamma_X) \otimes (\Gamma_Y \gamma_\alpha \gamma_\mu) \right] \end{aligned} \quad (91)$$

for $\Gamma_X = P_R$ or P_L , therefore no Fierz transformation is necessary to simplify the spinor structure of the total contribution. Finally we obtain

$$I_{PD}^a + I_{PD}^b + I_{PD}^c = g^2 \frac{N+1}{2N} \varepsilon^{ijk} K [\Gamma_X \otimes \Gamma_Y] [3T + A_{SP} + 2A_{VA}]. \quad (92)$$

Compared with the tree level result of eq.(83) we find that the vertex correction is multiplicative up to the one-loop level:

$$\Lambda_{PD} = \left[1 + g^2 \frac{N+1}{2N} \{3 \langle T \rangle + \langle A_{SP} \rangle + 2 \langle A_{VA} \rangle\} \right] \Lambda_{PD}^{(0)}, \quad (93)$$

where $\langle X \rangle$ ($X = T, A_{VA}, A_{SP}$) are defined in eq.(54). We remark that in the Wilson case \mathcal{O}_{PD} mixes with other operators which have different chiral structures under renormalization [10].

Taking account of the contribution of the wave function the lattice renormalization factor for \mathcal{O}_{PD} is expressed as

$$Z_{PD}^{lat} = (1 - w_0^2)^{3/2} Z_w^{3/2} Z_2^{3/2} V_{PD}, \quad (94)$$

where

$$V_{PD} = 1 + \frac{g^2}{16\pi^2} \left[-\delta_{PD} \log(\lambda a)^2 + v_{PD} \right], \quad (95)$$

$$v_{PD} = \frac{16\pi^2(N+1)}{2N} [3 \langle T \rangle + \langle\langle A_{SP} \rangle\rangle + 2 \langle\langle A_{VA} \rangle\rangle] + \delta_{PD} \log \pi^2 \quad (96)$$

with $\delta_{PD} = \frac{6(N+1)}{2N}$. Numerical values for v_{PD} , evaluated as before, are given in Table II as a function of M .

The corresponding continuum renormalization factors in the $\overline{\text{MS}}$ scheme are calculated employing the DRED scheme as the regularization in the Feynman gauge with the fictitious gluon mass λ . The vertex correction for \mathcal{O}_{PD} is

$$V_{PD}^{\overline{\text{MS}}} = 1 + \frac{g^2}{16\pi^2} \delta_{PD} \left[-\log(\lambda/\mu)^2 + 1 \right], \quad (97)$$

giving $v_{PD}^{\overline{\text{MS}}} = \delta_{PD}$. We remark that in this case the evanescent operator does not appear at the one-loop level.

Combining this result with the previous lattice one we finally obtain the relation between the operators $\mathcal{O}_{PD}^{\overline{\text{MS}}}$ and $\mathcal{O}_{PD}^{\text{lat}}$:

$$\mathcal{O}_{PD}^{\overline{\text{MS}}}(\mu) = \frac{1}{(1 - w_0^2)^{3/2} Z_w^{3/2}} Z_{PD}(\mu a) \mathcal{O}_{PD}^{\text{lat}}(1/a), \quad (98)$$

with

$$\begin{aligned} Z_{PD}(\mu a) &= \frac{(Z_2^{\overline{\text{MS}}})^{3/2} V_{PD}^{\overline{\text{MS}}}}{(Z_2)^{3/2} V_{PD}} \\ &= 1 + \frac{g^2}{16\pi^2} \left[(\delta_{PD} - 3C_F/2) \log(\mu a)^2 + z_{PD} \right], \end{aligned} \quad (99)$$

$$z_{PD} = v_{PD}^{\overline{\text{MS}}} - v_{PD} + \frac{3}{2} C_F \{ \Sigma_1^{\overline{\text{MS}}} - \Sigma_1 \}. \quad (100)$$

We present numerical values for z_{PD} in Table III and those for the mean-field improved one, z_{PD}^{MF} , in Table IV.

V. RENORMALIZATION FACTOR FOR B_K

As an application of results in the previous sections, we estimate a renormalization factor for the kaon B parameter B_K , defined by

$$B_K = \frac{\langle \overline{K}^0 | \mathcal{O}_+ | K^0 \rangle}{\frac{8}{3} \langle \overline{K}^0 | A_4 | 0 \rangle \langle 0 | A_4 | K^0 \rangle} \quad (101)$$

with $q_1 = q_3 = s$ and $q_2 = q_4 = d$ in \mathcal{O}_+ .

Denoting the renormalization factor between the continuum B_K at scale μ and the lattice one at scale $1/a$ as $Z_{B_K}(\mu a)$, we obtain

$$Z_{B_K}(\mu a) = \frac{(1 - w_0^2)^{-2} Z_w^{-2} Z_+(\mu a)}{(1 - w_0)^{-2} Z_w^{-2} Z_A(\mu a)^2} = \frac{Z_+(\mu a)}{Z_A(\mu a)^2}, \quad (102)$$

where

$$Z_+(\mu a) = 1 + \frac{g^2}{16\pi^2} [-4 \log(\mu a) + z_+] \quad (103)$$

from eq. (77) in this paper, and

$$Z_A(\mu a) = 1 + \frac{g^2 C_F}{16\pi^2} z_A \quad (104)$$

from Ref. [7], so that

$$Z_{B_K}(\mu a) = 1 + \frac{g^2}{16\pi^2} [-4\log(\mu a) + z_+ - 2C_F z_A]. \quad (105)$$

Note that z_A in Ref. [7] is evaluated in the NDR scheme while the DRED scheme is used for z_+ in this paper. From the result in Appendix B we have

$$z_A(\text{DRED}) = z_A(\text{NDR}) + 1/2, \quad z_+(\text{DRED}) = z_+(\text{NDR}) + 3. \quad (106)$$

In Ref. [3] B_K has been evaluated at $\beta = 5.85, 6.0$ with $M = 1.7$ and $\beta = 6.3$ with $M = 1.5$, using domain-wall QCD with the quenched approximation. Here we explicitly calculate $Z_{B_K}(\mu a)$ for these parameters. From Table III and the previous result [7], $z_+ = -41.854(-42.399)$, $z_A = -17.039(-16.827)$ and $z_+ - 2C_F z_A = 3.583(2.473)$ for $M = 1.7(1.5)$ in the DRED scheme, and $z_+ = -44.854(-45.399)$, $z_A = -17.539(-17.327)$ and $z_+ - 2C_F z_A = 1.917(0.8063)$ for $M = 1.7(1.5)$ in the NDR scheme. Taking $\mu = 1/a$ and $g^2 = g_{\overline{MS}}^2(1/a)$, estimated by the formula

$$\frac{1}{g_{\overline{MS}}^2}(1/a) = P \frac{\beta}{6} - 0.13486 \quad (107)$$

for the quenched QCD with P being the average value of the plaquette, we have $Z_{B_K} = 1.053$ (1.029), 1.049 (1.026) and 1.030 (1.010) at $\beta = 5.85, 6.0$ and 6.3, respectively, in the DRED (NDR) scheme. Sizes of one-loop corrections for B_K are not so large, 1–5%, at these β values even without mean-field improvement, since the large contribution, which comes from a $(1 - w_0)Z_w$ factor, cancels out in the ratio of \mathcal{O}_+ and A_4^2 .

If we employ the mean-field improvement by replacing $M \rightarrow \widetilde{M} = M + 4(u - 1)$ with $u = P^{1/4}$, we obtain $Z_{B_K} = 1.018$ (0.994), 1.017 (0.994) and 1.009 (0.988) at $\beta = 5.85, 6.0$ and 6.3, respectively, in the DRED (NDR) scheme. See Appendix A for some remarks.

Note that there is no mean-field improvement factor for B_K in actual simulations since, as mentioned before, it is defined by the ratio. Therefore the difference between values of Z_{B_K} with and without mean-field improvement comes from higher order ambiguity in perturbation theory.

Necessary informations for the analysis in this section are given in Table V, together with values of Z_{B_K} .

VI. CONCLUSION

In this paper we have calculated the one-loop contributions for the renormalization factors of the three- and four-quark operators in DWQCD. We have demonstrated that the three- and four-quark operators in DWQCD can be renormalized without any operator mixing between different chiralities as opposed to the Wilson case. This desirable property in DWQCD would practically surpass the cost of the introduction of an unphysical fifth dimension. The numerical values for the finite parts z_X with $X = \pm, 1, 2, PD$ settle in reasonable magnitude with the mean-field improvement, while unimproved values are rather large in general.

In this work we do not treat the operators which yield the so-called “penguin” diagram. It seems feasible to carry out the calculation of their renormalization factors, which we leave to future investigation.

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APPENDIX A: MEAN-FIELD IMPROVEMENT

The mean-field improvement [11] in our paper uses

$$u = 1 - \frac{g^2 C_F}{2} T$$

with $T = 0.15493$, which is the value for the link in Feynman gauge. It may be better to use u from K_c or plaquette in DWQCD. In that case

$$u = 1 - \frac{g^2 C_F}{2} (T + \delta T),$$

where $\delta T = 0.00793$ for K_c , or $\delta T = -0.02993$ for plaquette. Accordingly we have to modify renormalization factors as follows:

$$z_w^{MF}(T + \delta T) = z_w^{MF}(T) + \frac{2w_0}{1 - w_0^2} 16\pi^2 \times 2\delta T, \quad (108)$$

$$z_2^{MF}(T + \delta T) = z_2^{MF}(T) + 16\pi^2 \times \delta T/2, \quad (109)$$

$$z_{\Gamma, \text{bilinear}}^{MF}(T + \delta T) = z_{\Gamma, \text{bilinear}}^{MF}(T) + 16\pi^2 \times \delta T/2, \quad (110)$$

$$z_{\Gamma, 4\text{-quark}}^{MF}(T + \delta T) = z_{\Gamma, 4\text{-quark}}^{MF}(T) + 16\pi^2 \times \delta T/2 \times 2C_F, \quad (111)$$

$$z_{\Gamma, 3\text{-quark}}^{MF}(T + \delta T) = z_{\Gamma, 3\text{-quark}}^{MF}(T) + 16\pi^2 \times \delta T/2 \times \frac{3}{2}C_F. \quad (112)$$

APPENDIX B: NAIVE DIMENSIONAL REGULARIZATION(NDR)

In this appendix we compile the finite part of the renormalization constant in the $\overline{\text{MS}}$ subtraction scheme with the Naive Dimensional Regularization:

$$\Sigma_1^{\overline{\text{MS}}} = 1/2, \quad (113)$$

$$v_+^{\overline{\text{MS}}} = \delta_+ \times \left\{ 3/2 - \frac{2N+3}{N-2} \right\}, \quad (114)$$

$$v_-^{\overline{\text{MS}}} = \delta_- \times \left\{ 3/2 - \frac{2N-3}{N+2} \right\}, \quad (115)$$

$$v_{ij}^{\overline{\text{MS}}} = \begin{pmatrix} \delta_1 \times \left\{ 3/2 - \frac{2(N^2-5)}{N^2-4} \right\}, & \delta_2 \times 3/4 \\ \frac{1}{N} \times 3, & \delta_2 \times 1/2 \end{pmatrix}, \quad (116)$$

$$v_{PD}^{\overline{\text{MS}}} = \delta_{PD} \times 2/3, \quad (117)$$

where v_{ij} with $i, j = 1, 2$ is a matrix, which represents the mixing of the finite part for $\mathcal{O}_{1,2}$. The one-loop vertex corrections for \mathcal{O}_Γ ($\Gamma = \pm, 1, 2$) require to specify their evanescent operators, which originates from the property that the Fierz transformation can not be defined in the NDR scheme. We employ

$$E_\pm^{\text{NDR}} = \gamma_\rho \gamma_\delta \gamma_\mu (1 - \gamma_5) \otimes \gamma_\mu (1 - \gamma_5) \gamma_\delta \gamma_\rho - (2 - D)^2 \gamma_\mu (1 - \gamma_5) \otimes \gamma_\mu (1 - \gamma_5), \quad (118)$$

$$E_{1,2}^{\text{NDR}} = \gamma_\rho \gamma_\delta \gamma_\mu (1 - \gamma_5) \otimes \gamma_\mu (1 + \gamma_5) \gamma_\delta \gamma_\rho - D^2 \gamma_\mu (1 - \gamma_5) \otimes \gamma_\mu (1 + \gamma_5), \quad (119)$$

where D is the reduced space-time dimension. On the other hand, the evanescent operator does not appear in the one-loop vertex correction of \mathcal{O}_{PD} .

For later convenience, values of the finite part of quark bilinear operators are also given here. For NDR scheme

$$z_{V,A}^{\overline{\text{MS}}} = 0, \quad z_{S,P}^{\overline{\text{MS}}} = 5/2, \quad z_T^{\overline{\text{MS}}} = 1/2, \quad (120)$$

while for DRED scheme

$$z_{V,A}^{\overline{\text{MS}}} = 1/2, \quad z_{S,P}^{\overline{\text{MS}}} = 7/2, \quad z_T^{\overline{\text{MS}}} = -1/2, \quad (121)$$

where the evanescent operators are

$$E_{\gamma_\mu}^{\text{DRED}} = \bar{\delta}_{\mu\nu} \gamma_\nu - \frac{D}{4} \gamma_\mu, \quad (122)$$

$$E_{\gamma_\mu \gamma_5}^{\text{DRED}} = \bar{\delta}_{\mu\nu} \gamma_\nu \gamma_5 - \frac{D}{4} \gamma_\mu \gamma_5. \quad (123)$$

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TABLES

TABLE I. Color factors for $I_\Gamma^{a,b,c}$ ($\Gamma = \pm, 1, 2$).

Γ	J_a^{AB}	J_b^{AB}	J_c^{AB}
\pm	$T^A T^B \tilde{\otimes} 1 \pm T^A \tilde{\odot} T^B$	$T^A \tilde{\otimes} T^B \pm T^A \tilde{\odot} T^B$	$T^A \tilde{\otimes} T^B \pm T^A T^B \tilde{\odot} 1$
1	$-NT^A T^B \tilde{\otimes} 1 + T^A \tilde{\odot} T^B$	$-NT^A \tilde{\otimes} T^B + T^A \tilde{\odot} T^B$	$-NT^A \tilde{\otimes} T^B + T^A T^B \tilde{\odot} 1$
2	$T^A \tilde{\odot} T^B$	$T^A \tilde{\odot} T^B$	$T^A T^B \tilde{\odot} 1$

TABLE II. Numerical values for V_Γ ($\Gamma = \pm, 1, 2, PD$) as a function of M .

M	V_+	V_-	V_1	V_2	V_{PD}
0.05	13.9096(8)	10.847(8)	13.3992(19)	8.805(12)	8.646(5)
0.10	13.5696	11.537	13.2309	10.182	8.992
0.15	13.2941	12.098	13.0948	11.301	9.273
0.20	13.0548	12.587	12.9768	12.275	9.518
0.25	12.8391	13.029	12.8708	13.155	9.740
0.30	12.6404	13.438	12.7734	13.970	9.946
0.35	12.4542	13.822	12.6822	14.734	10.139
0.40	12.2775	14.188	12.5960	15.462	10.323
0.45	12.1083	14.539	12.5135	16.160	10.499
0.50	11.9449	14.880	12.4341	16.837	10.671
0.55	11.7861	15.212	12.3571	17.496	10.838
0.60	11.6307	15.538	12.2819	18.143	11.002
0.65	11.4779	15.859	12.2081	18.780	11.164
0.70	11.3269	16.178	12.1354	19.412	11.325
0.75	11.1770	16.495	12.0634	20.041	11.485
0.80	11.0275	16.813	11.9917	20.670	11.645
0.85	10.8779	17.131	11.9201	21.300	11.806
0.90	10.7276	17.452	11.8484	21.935	11.968
0.95	10.5760	17.777	11.7762	22.578	12.133
1.00	10.4225	18.107	11.7033	23.230	12.300
1.05	10.2659	18.437	11.6278	23.885	12.466
1.10	10.1076	18.790	11.5547	24.579	12.646
1.15	9.9443	19.139	11.4768	25.269	12.822
1.20	9.7768	19.505	11.3981	25.990	13.007
1.25	9.6037	19.880	11.3165	26.732	13.198
1.30	9.4244	20.272	11.2323	27.504	13.396
1.35	9.2375	20.680	11.1446	28.309	13.603
1.40	9.0419	21.109	11.0530	29.153	13.820
1.45	8.8361	21.560	10.9567	30.042	14.049
1.50	8.6183	22.037	10.8547	30.983	14.291
1.55	8.3863	22.547	10.7464	31.987	14.550
1.60	8.1375	23.093	10.6301	33.063	14.827
1.65	7.8685	23.683	10.5043	34.227	15.127
1.70	7.5747	24.328	10.3668	35.496	15.454
1.75	7.2502	25.038	10.2149	36.897	15.814
1.80	6.8864	25.832	10.0440	38.463	16.216
1.85	6.4706	26.737	9.8482	40.247	16.675
1.90	5.9812	27.794	9.6168	42.337	17.210
1.95	5.3749	29.093	9.3281	44.907	17.867

TABLE III. Numerical values for z_Γ ($\Gamma = \pm, 1, 2, PD$) as a function of M .

M	z_+	z_-	z_1	z_2	z_{PD}
0.05	-49.908(10)	-40.845(11)	-48.397(8)	-34.803(15)	-32.144(7)
0.10	-49.332	-41.300	-47.994	-35.945	-32.314
0.15	-48.847	-41.651	-47.647	-36.853	-32.437
0.20	-48.416	-41.949	-47.338	-37.637	-32.539
0.25	-48.025	-42.215	-47.057	-38.341	-32.629
0.30	-47.663	-42.461	-46.796	-38.993	-32.713
0.35	-47.325	-42.693	-46.553	-39.605	-32.792
0.40	-47.007	-42.918	-46.326	-40.191	-32.870
0.45	-46.706	-43.137	-46.111	-40.758	-32.948
0.50	-46.419	-43.354	-45.908	-41.311	-33.027
0.55	-46.145	-43.571	-45.716	-41.855	-33.108
0.60	-45.883	-43.790	-45.534	-42.395	-33.191
0.65	-45.631	-44.012	-45.361	-42.933	-33.279
0.70	-45.388	-44.239	-45.196	-43.473	-33.370
0.75	-45.153	-44.472	-45.040	-44.017	-33.467
0.80	-44.927	-44.712	-44.891	-44.569	-33.570
0.85	-44.708	-44.961	-44.750	-45.130	-33.679
0.90	-44.496	-45.220	-44.617	-45.704	-33.795
0.95	-44.290	-45.491	-44.490	-46.292	-33.918
1.00	-44.091	-45.776	-44.372	-46.899	-34.051
1.05	-43.898	-46.070	-44.260	-47.517	-34.190
1.10	-43.710	-46.392	-44.157	-48.181	-34.347
1.15	-43.528	-46.723	-44.061	-48.853	-34.510
1.20	-43.352	-47.079	-43.973	-49.565	-34.688
1.25	-43.180	-47.457	-43.893	-50.308	-34.880
1.30	-43.014	-47.862	-43.822	-51.094	-35.088
1.35	-42.853	-48.296	-43.760	-51.924	-35.315
1.40	-42.697	-48.763	-43.708	-52.808	-35.561
1.45	-42.545	-49.269	-43.666	-53.751	-35.831
1.50	-42.399	-49.818	-43.635	-54.763	-36.127
1.55	-42.257	-50.417	-43.617	-55.857	-36.453
1.60	-42.119	-51.074	-43.611	-57.044	-36.813
1.65	-41.985	-51.800	-43.621	-58.343	-37.214
1.70	-41.854	-52.607	-43.646	-59.776	-37.663
1.75	-41.725	-53.513	-43.690	-61.372	-38.170
1.80	-41.595	-54.541	-43.753	-63.172	-38.748
1.85	-41.460	-55.727	-43.838	-65.237	-39.417
1.90	-41.311	-57.124	-43.946	-67.666	-40.207
1.95	-41.121	-58.840	-44.074	-70.652	-41.177

TABLE IV. Numerical value for z_{Γ}^{MF} ($\Gamma = \pm, 1, 2, PD$) as a function of M .

M	z_{+}^{MF}	z_{-}^{MF}	z_1^{MF}	z_2^{MF}	z_{PD}^{MF}
0.05	-17.287(10)	-8.224(11)	-15.776(8)	-2.182(15)	-7.679(7)
0.10	-16.712	-8.679	-15.373	-3.324	-7.848
0.15	-16.226	-9.030	-15.027	-4.232	-7.972
0.20	-15.796	-9.328	-14.718	-5.016	-8.074
0.25	-15.404	-9.594	-14.436	-5.720	-8.164
0.30	-15.042	-9.840	-14.175	-6.372	-8.247
0.35	-14.705	-10.073	-13.933	-6.98	-8.326
0.40	-14.386	-10.297	-13.705	-7.57	-8.404
0.45	-14.085	-10.516	-13.490	-8.13	-8.482
0.50	-13.799	-10.734	-13.288	-8.69	-8.561
0.55	-13.525	-10.951	-13.096	-9.23	-8.642
0.60	-13.262	-11.169	-12.913	-9.77	-8.726
0.65	-13.010	-11.391	-12.740	-10.3	-8.813
0.70	-12.767	-11.618	-12.575	-10.8	-8.905
0.75	-12.532	-11.851	-12.419	-11.3	-9.002
0.80	-12.306	-12.091	-12.270	-11.9	-9.104
0.85	-12.087	-12.340	-12.129	-12.5	-9.213
0.90	-11.875	-12.600	-11.996	-13.0	-9.329
0.95	-11.670	-12.871	-11.870	-13.6	-9.453
1.00	-11.470	-13.155	-11.751	-14.2	-9.585
1.05	-11.278	-13.449	-11.639	-14.8	-9.725
1.10	-11.089	-13.772	-11.536	-15.5	-9.882
1.15	-10.908	-14.103	-11.440	-16.2	-10.044
1.20	-10.731	-14.459	-11.352	-16.9	-10.223
1.25	-10.559	-14.836	-11.272	-17.6	-10.414
1.30	-10.393	-15.241	-11.201	-18.4	-10.623
1.35	-10.232	-15.675	-11.139	-19.3	-10.849
1.40	-10.076	-16.143	-11.087	-20.1	-11.096
1.45	-9.925	-16.648	-11.045	-21.13	-11.365
1.50	-9.778	-17.197	-11.014	-22.14	-11.661
1.55	-9.636	-17.796	-10.996	-23.23	-11.987
1.60	-9.498	-18.453	-10.991	-24.42	-12.347
1.65	-9.364	-19.179	-11.000	-25.72	-12.749
1.70	-9.233	-19.986	-11.025	-27.15	-13.198
1.75	-9.104	-20.892	-11.069	-28.75	-13.705
1.80	-8.975	-21.920	-11.132	-30.55	-14.283
1.85	-8.840	-23.106	-11.217	-32.61	-14.952
1.90	-8.690	-24.503	-11.326	-35.04	-15.742
1.95	-8.500	-26.219	-11.453	-38.03	-16.711

TABLE V. Renormalization factor for $B_K(1/a)$ at some parameters.

β	5.85		6.0		6.3	
M	1.70		1.70		1.50	
P	0.57506		0.59374		0.62246	
$g_{MS}^2(1/a)$	2.3484		2.1792		1.9278	
u	0.87082		0.87781		0.88823	
\widetilde{M}	1.20		1.20		1.05	
	DRED	NDR	DRED	NDR	DRED	NDR
z_+	-41.854	-44.854	-41.854	-44.854	-42.399	-45.399
z_A	-17.039	-17.539	-17.039	-17.539	-16.827	-17.327
$z_+ - 2C_F z_A$	3.583	1.917	3.583	1.917	2.473	0.806
$Z_{B_K}(\mu a = 1)$	1.053	1.029	1.049	1.026	1.030	1.010
z_+^{MF}	-17.033	-20.033	-17.033	-20.033	-17.580	-20.580
z_A^{MF}	-6.853	-7.353	-6.853	-7.353	-6.864	-7.364
$z_+^{MF} - 2C_F z_A^{MF}$	1.242	-0.425	1.242	-0.425	0.724	-0.943
$Z_{B_K}^{MF}(\mu a = 1)$	1.018	0.994	1.017	0.994	1.009	0.988

FIGURES

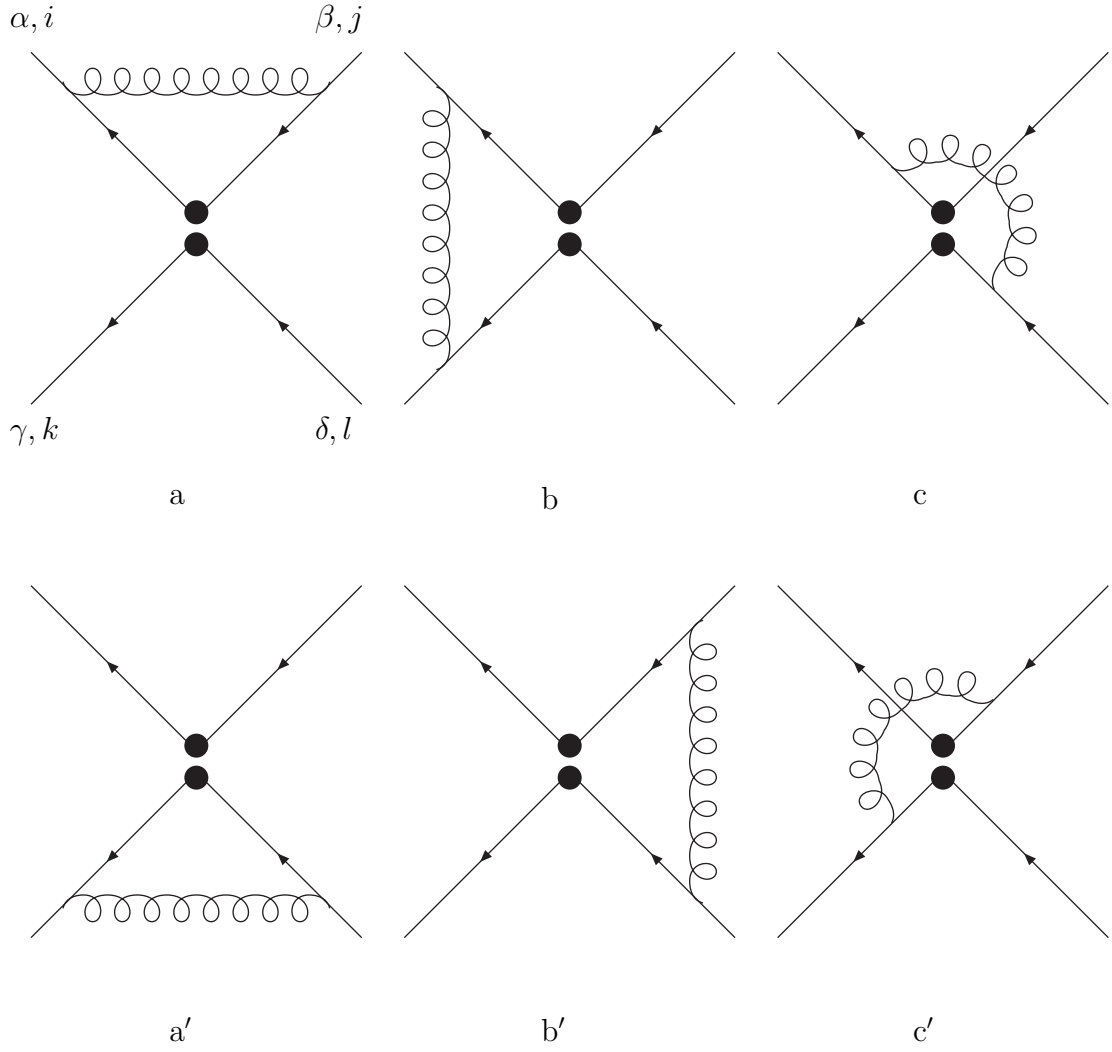


FIG. 1. One-loop vertex corrections for the four-quark operator. $\alpha, \beta, \gamma, \delta$ and i, j, k, l label Dirac and color indices respectively.

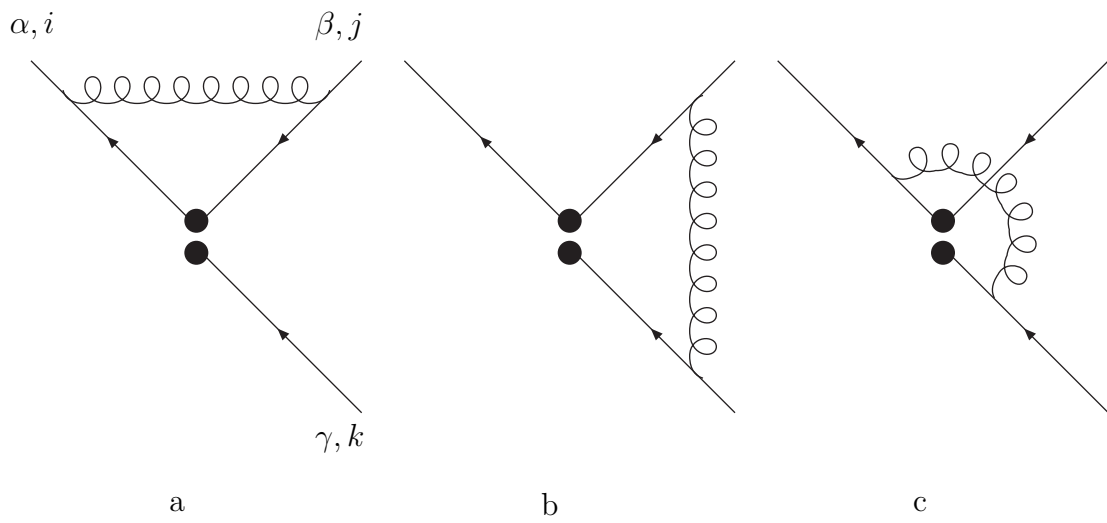


FIG. 2. One-loop vertex corrections for the three-quark operator. α, β, γ and i, j, k label Dirac and color indices respectively.